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BOUNDARY LIMITS AND NON-INTEGRABILITY OF $\mathcal{M}$-SUBHARMONIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{C}^n$ $(n \geq 1)$

MANFRED STOLL

Abstract. In this paper we consider weighted non-tangential and tangential boundary limits of non-negative functions on the unit ball B in $\mathbb{C}^n$ that are subharmonic with respect to the Laplace-Beltrami operator $\widetilde{\Delta}$ on B. Since the operator $\widetilde{\Delta}$ is invariant under the group $\mathcal{M}$ of holomorphic automorphisms of B, functions that are subharmonic with respect to $\widetilde{\Delta}$ are usually referred to as $\mathcal{M}$-subharmonic functions. Our main result is as follows: Let f be a non-negative $\mathcal{M}$-subharmonic function on B satisfying

$$\int_B (1-|z|^2)^\gamma f^p(z)\, d\lambda(z) < \infty$$

for some $p > 0$ and some $\gamma > \min\{n, pn\}$, where λ is the $\mathcal{M}$-invariant measure on B. Suppose $\tau \geq 1$. Then for a.e. $\zeta \in S$,

$$f^p(z) = o\left((1-|z|^2)^{n/\tau-\gamma}\right)$$

uniformly as $z \to \zeta$ in each $\mathcal{T}_{\tau,\alpha}(\zeta)$, where for $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$)

$$\mathcal{T}_{\tau,\alpha}(\zeta) = \{z \in B : |1 - \langle z, \zeta\rangle|^\tau < \alpha(1-|z|^2)\}.$$

We also prove that for $\gamma \leq \min\{n, pn\}$ the only non-negative $\mathcal{M}$-subharmonic function satisfying the above integrability criteria is the zero function.

Introduction

The results of this paper were motivated by the following result of F. W. Gehring [GE] (see also [TS, Theorem IV. 41]):

Theorem A. *Suppose $w(z)$ is a non-negative subharmonic function in the unit disc $|z| < 1$ in $\mathbb{C}$ satisfying*

$$\iint_{|z|<1} w^p(z)\, dx\, dy < \infty, \quad z = x + iy, \tag{1.1}$$

for some $p > 1$. Then for almost every θ,

$$w(z) = o\left((1-|z|^2)^{-1/p}\right)$$

uniformly as $z \to e^{i\theta}$ in each non-tangential approach region $\Gamma_\alpha(e^{i\theta})$.

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This last statement is equivalent to

$$\lim_{r\to 1} \sup_{\substack{z\in\Gamma_\alpha(e^{i\theta})\\ |z|\ge r}} (1-|z|^2)w^p(z) = 0$$

for almost every θ, where for $\alpha > \frac{1}{2}$,

$$\Gamma_\alpha(e^{i\theta}) = \{z : |e^{i\theta} - z| < \alpha(1-|z|^2),\ |z| < 1\}. \tag{1.2}$$

The proof of Theorem A used the Hardy-Littlewood theorem, which accounts for the assumption that $p > 1$.

Using techniques of potential theory, we extend the result of Gehring in several directions. First, we remove the restriction $p > 1$ and prove that Theorem A is valid for all p, $0 < p < \infty$. Second, in addition to non-tangential limits, we will also consider weighted boundary limits along tangential approach regions. Finally, since our methods are equally valid in the unit ball B in $\mathbb{C}^n$, we will state and prove the result for functions that are subharmonic with respect to the Laplace-Beltrami operator or invariant Laplacian $\widetilde{\Delta}$ on B. When $n = 1$, this is equivalent to the usual definition of a subharmonic function.

Prior to stating the main result of the paper we first introduce some notation. Let B denote the unit ball in $\mathbb{C}^n$ with boundary S, $\mathcal{M}$ the group of holomorphic automorphisms of B, and λ the $\mathcal{M}-$invariant volume measure on B. Functions that are harmonic or subharmonic with respect to the Laplace-Beltrami operator $\widetilde{\Delta}$ on B are usually referred to as $\mathcal{M}$-harmonic and $\mathcal{M}$-subharmonic functions, or also as invariant harmonic and invariant subharmonic functions.

Let $\zeta \in S$. For $\tau \ge 1$ and $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$), set

$$\mathcal{T}_{\tau,\alpha}(\zeta) = \{z \in B : |1 - \langle z, \zeta\rangle|^\tau < \alpha(1-|z|^2)\}. \tag{1.3}$$

When $\tau = 1$ (and $\alpha > \frac{1}{2}$) these are the admissible approach regions of Koranyi for $n > 1$, and the non-tangential regions Γ_α when $n = 1$. When $n > 1$, the regions $\mathcal{T}_{1,\alpha}(\zeta)$ provide non-tangential approach to ζ in the complex normal direction, but parabolic approach in the complex tangential direction. For $\tau > 1$, the regions $\mathcal{T}_{\tau,\alpha}(\zeta)$ have tangential contact in all directions at ζ. For example, when $n = 1$, $\mathcal{T}_{2,1}(\zeta)$ is the disc of radius $\frac{1}{2}$ with center $\frac{1}{2}\zeta$. For $n > 1$, $\mathcal{T}_{2,\alpha}(\zeta)$ is an ellipsoid. With $\zeta = e_1 = (1, 0, ..., 0)$,

$$\mathcal{T}_{2,\alpha}(e_1) = \{(z_1, z') \in B : \frac{|z_1 - \frac{1}{1+\alpha}|^2}{\beta^2} + \frac{|z'|^2}{\beta} < 1\},$$

where $\beta = \alpha/(1+\alpha)$ [RU, p. 175].

The main result of the paper is as follows:

Theorem B. *Let f be a non-negative $\mathcal{M}$-subharmonic function on B satisfying*

$$\int_B (1-|z|^2)^\gamma f^p(z)\, d\lambda(z) < \infty \tag{1.4}$$

for some $p > 0$ and $\gamma > \min\{n, pn\}$. Then for each $\tau \ge 1$ and $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$)

$$\lim_{\rho\to 1} \sup_{z\in\mathcal{T}_{\tau,\alpha,\rho}(\zeta)} (1-|z|^2)^{\gamma - n/\tau} f^p(z) = 0 \quad \textit{for a.e. } \zeta \in S,$$

where $\mathcal{T}_{\tau,\alpha,\rho}(\zeta) = \{z \in \mathcal{T}_{\tau,\alpha}(\zeta) : \rho \le |z| < 1\}$.

When $n = 1$, the $\mathcal{M}$-invariant measure λ is given by $d\lambda(z) = (1 - |z|^2)^{-2} dx\, dy$. Thus with $\gamma = 2$ and $\tau = 1$ one obtains Theorem A for all p, $0 < p < \infty$. Although Theorem B is stated as an almost everywhere result, the result we will prove (Theorem 3.1) will be stated in terms of of the $s-$dimensional $(0 < s \le n)$ "non-isotropic" Hausdorff capacity or measure on S. In Theorem 3.4 we will also investigate the rate of growth of the integral means of f^p.

Since every plurisubharmonic function on B is also $\mathcal{M}$-subharmonic, our results are also valid for non-negative plurisubharmonic functions on B. In particular, for holomorphic functions on B we have the following result, which as the special case $n = 1$, $\tau = 1$, and $\gamma = 2$ includes Theorem 2 of the paper by Gehring ([GE]).

Theorem C. *Let f be a holomorphic function on B for which $|f|$ satisfies (1.4) for some $p > 0$ and $\gamma > n$. Suppose $\tau \ge 1$ and $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$). Then for almost every $\zeta \in S$,*

$$f(z) = o\left((1 - |z|^2)^{(n-\gamma\tau)/p\tau}\right)$$

uniformly as $z \to \zeta$ in $\mathcal{T}_{\tau,\alpha}(\zeta)$.

The second main result of the paper concerns the following: Given $0 < p < \infty$, for what values of γ does there exist a non-negative $\mathcal{M}$-subharmonic function f on B, $f \not\equiv 0$, such that the integral in (1.4) is finite? For a holomorphic function f on B, it is easily shown (see the Remark following the proof of Theorem 4.1) that γ must be greater than n. Specifically, if f is holomorphic on B and satisfies $\int_B (1 - |z|^2)^\gamma |f(z)|^p \, d\lambda(z) < \infty$ for some $p > 0$ and $\gamma \le n$, then $f(z) = 0$ for all $z \in B$. For $\mathcal{M}$-subharmonic functions this is still the case if $p \ge 1$. When $0 < p < 1$, then, as we will see in Section 4, there exist values of $\gamma \le n$ and non-negative $\mathcal{M}$-subharmonic functions on B such that the integral in (1.4) is finite. However, in Theorem 4.1 we prove that if $0 < p < 1$ and $\gamma \le pn$, then the only non-negative $\mathcal{M}$-subharmonic function f satisfying (1.4) is the zero function. This accounts for the assumption $\gamma > \min\{n, pn\}$ in the hypothesis of Theorem B. By example it will be shown that at least when $n = 1$, this result is sharp. In this section we also consider the integrability of $\mathcal{M}$-subharmonic functions and non-negative $\mathcal{M}$-harmonic functions. In Theorem 4.2 we prove that if $0 < p < 1$, and h is a non-negative $\mathcal{M}$-harmonic function satisfying (1.4) for some $\gamma \le \max\{pn, (1-p)n\}$, then $h \equiv 0$. By example it will be shown that this is sharp.

Tangential boundary limits of holomorphic functions and $\mathcal{M}$-subharmonic functions in both the unit disc and unit ball of $\mathbb{C}^n$ have been considered by others. Many of the results however involve the existence of pointwise boundary limits of functions in Dirichlet-type spaces or of Green potentials. In [KI2], J. R. Kinney considered tangential boundary limits of analytic functions $f(z) = \sum_{n \ge 0} a_n z^n$ in the unit disc satisfying $\sum_{n \ge 0} n^\alpha |a_n|^2 < \infty$ for some α, $0 < \alpha \le 1$. This is easily shown to be equivalent to f satisfying

$$\iint_{|z|<1} (1 - |z|^2)^{1-\alpha} |f'(z)|^2 \, dxdy < \infty.$$

The special case $\alpha = 1$ gives the usual Dirichlet space $\mathcal{D}$. The results of Kinney imply that every $f \in \mathcal{D}$ has $\mathcal{T}_\tau$-limits almost everywhere on $|z| = 1$ for every $\tau \ge 1$, and also contain information about the capacities of exceptional sets. The results of Kinney have been extended by J. R. Twomey [TW] to include weights more general than n^α. Tangential boundary limits of functions in Dirichlet-type spaces have also been considered by A. Nagel, W. Rudin, and J. H. Shapiro [NRS] for the upper half-space $\mathbb{R}^{n+1}_+$, and by L. Rzepecki [RZ] and J. Sueiro [SU] for the unit ball B in $\mathbb{C}^n$.

In a different direction, tangential boundary limits of Blaschke products have been studied by G. T. Cargo [CA] and J. R. Kinney [KI1], among others. These results have been extended by the author to invariant Green potentials in the unit ball of $\mathbb{C}^n$ in [ST1] and [ST3]. The existence of tangential boundary limits of $\mathcal{M}$-harmonic Besov functions has been studied by K. T. Hahn and E. H. Youssfi in [HY], and for $\mathcal{M}$-subharmonic functions in Dirichlet-type spaces by K. T. Hahn, E. H. Youssfi, and the author in [HSY]. Tangential boundary limits of harmonic functions and Green potentials have also been considered by many authors for domains in $\mathbb{R}^n$ $(n \geq 3)$. A good reference for results in this direction is the paper by R. D. Berman and W. S. Cohn [BC]. Also, the question of integrability of non-negative subharmonic functions on domains in $\mathbb{R}^n$ has previously been considered by N. Suzuki in [SZ]. Weighted L^p-integrability of non-negative $\mathcal{M}$-superharmonic functions on B has also been considered by S. Zhao in [ZH].

2. Notation and preliminary results

As in the Introduction let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the unit ball in $\mathbb{C}^n$ with boundary S. Following the notation of [RU], let $d\nu$ and $d\sigma$ denote normalized Lebesgue measure on B and S respectively. For each $a \in B$, let $\varphi_a(z)$ denote the involutive automorphism of B satisfying $\varphi_a(a) = 0$, $\varphi_a(0) = a$, and $\varphi_a(\varphi_a(z)) = z$. By [RU, p.26],

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}. \tag{2.1}$$

Let $\mathcal{M}$ denote the group of holomorphic automorphisms of B. Then any $\psi \in \mathcal{M}$ has a unique representation $\psi = U \circ \varphi_a$ for some $a \in B$ and $U \in \mathbf{U}(n)$, the group of unitary transformations of $\mathbb{C}^n$. Each $\psi \in \mathcal{M}$ is continuous on $\overline{B}$ with $\psi(S) = S$. Let λ be the measure on B defined by

$$d\lambda(z) = \frac{d\nu(z)}{(1-|z|^2)^{n+1}}\,.$$

Then λ is invariant under $\mathcal{M}$; i.e. $\int_B f(z)\,d\lambda(z) = \int_B (f\circ\psi)(z)\,d\lambda(z)$ for each $f \in L^1(d\lambda)$ and all $\psi \in \mathcal{M}$.

An upper semicontinuous function $f : B \to [-\infty, \infty)$ with $f \not\equiv -\infty$ is $\mathcal{M}$-*subharmonic* or *invariant subharmonic* on B if for each $a \in B$,

$$f(a) \leq \int_S f(\varphi_a(rt))\,d\sigma(t), \quad 0 < r < 1. \tag{2.2}$$

If equality holds in (2.2), then f is called $\mathcal{M}$-*harmonic* or *invariant harmonic* on B. Also, f is $\mathcal{M}$-*superharmonic* if $-f$ is $\mathcal{M}$-subharmonic. For $f \in C^2(B)$, inequality (2.2) is equivalent to $\widetilde{\Delta} f \geq 0$, where

$$\widetilde{\Delta} f = \frac{4(1-|z|^2)}{n+1} \sum_{i,j=1}^{n} [\delta_{i,j} - \overline{z}_i z_j] \frac{\partial^2 f}{\partial z_j \partial \overline{z}_i}$$

is the Laplace-Beltrami operator or the invariant Laplacian on B. The operator $\widetilde{\Delta}$ is invariant under $\mathcal{M}$; i.e., $\widetilde{\Delta}(f\circ\psi) = (\widetilde{\Delta} f)\circ\psi$ for all $\psi \in \mathcal{M}$, $f \in C^2(B)$. When

$n = 1$,

$$\widetilde{\Delta} f = 2(1-|z|^2)^2 \frac{\partial^2 f}{\partial z \partial \overline{z}},$$

and thus a function f on the unit disc is $\mathcal{M}$-subharmonic if and only if f is subharmonic.

The *Green's function* for the operator $\widetilde{\Delta}$ is given by $G(z,w) = g(\varphi_z(w))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^{1} (1-t^2)^{n-1} t^{-2n+1}\, dt.$$

Also, the invariant *Poisson kernel* P on $B \times S$ is given by

$$P(z,t) = \frac{(1-|z|^2)^n}{|1-\langle z,t\rangle|^{2n}}, \quad z \in B,\, t \in S. \tag{2.3}$$

For $a \in B$, $0 < r < 1$, set

$$E(a,r) = \varphi_a(rB) = \{z \in B : |\varphi_a(z)| < r\}.$$

By (2.1), for $z \in E(a,r)$,

$$\left(\frac{1-r}{1+r}\right)(1-|a|^2) \le (1-|z|^2) \le \left(\frac{1+r}{1-r}\right)(1-|a|^2). \tag{2.4}$$

The following result, the proof of which may be found in [ST1, Lemma 1]; [ST2, Lemma 8.17] will be needed.

Lemma 2.1. *Let $0 < r < 1$, $\alpha > 0$ and $\zeta \in S$. If $a \in \mathcal{T}_{\tau,\alpha}(\zeta)$, $\tau \ge 1$, then*

$$\varphi_a(rB) \subset \mathcal{T}_{\tau,c}(\zeta) \quad \textit{for any } c \ge \alpha \left(\frac{1+r}{1-r}\right)^{\tau+1}.$$

The following inequality, the proof of which may be found in [PA, Theorem 2.1] or [ST2, Proposition 10.1], is crucial in the proof of the main results.

Lemma 2.2. *If f is a non-negative $\mathcal{M}$-subharmonic function on B, then for all p, $0 < p < \infty$, $a \in B$, and $0 < r < 1$,*

$$f^p(a) \le \frac{C(n,p,r)}{r^{2n}} \int_{E(a,r)} f^p(w)\, d\lambda(w) \tag{2.5}$$

where

$$C(n,p,r) = \begin{cases} (1-r^2)^n, & 1 \le p < \infty, \\ 2^{2n/p}, & 0 < p < 1. \end{cases}$$

Remark. For $p \ge 1$, inequality (2.5) is the invariant volume mean-value inequality for the $\mathcal{M}$-subharmonic function f^p. For $0 < p < 1$, the euclidean version of (2.5) for harmonic functions in the unit disc is essentially due to G. H. Hardy and J. E. Littlewood [HL] (see also [KO, p. 253]). For harmonic functions on domains in $\mathbb{R}^n$ the result is due to C. Fefferman and E. Stein [FS, p. 172].

Finally, for the statement and proof of the main result we introduce the concept of "non-isotropic" s-dimensional Hausdorff capacity or measure. For $\zeta \in S$, $\delta > 0$, let $Q(\zeta,\delta)$ denote the "non-isotropic" ball in S defined by

$$Q(\zeta,\delta) = \{\eta \in S : |1-\langle \eta,\zeta\rangle| < \delta\}.$$

As in [CO], if K is a compact subset of S, $0 < s \le n$, the "non-isotropic" s-dimensional Hausdorff capacity of K is defined by

$$H_s(K) = \inf \sum \delta_j^s \,,$$

where the infimum is over all covers $\{Q(\zeta_j, \delta_j)\}$ of K. If A is an arbitrary subset of S, then

$$H_s(A) = \sup\{H_s(K) : K \text{ compact } \subset A\}.$$

Since $\sigma(Q(\zeta,\delta)) \approx \delta^n$, when $s = n$, H_n is equivalent to Lebesgue measure on S. When $n = 1$, the "non-isotropic" Hausdorff capacity corresponds to the usual Hausdorff capacity on the boundary. The following result of W. S. Cohn will be needed for the proof of Theorem 3.1.

Lemma 2.3. [CO, Theorem 1] *For a compact subset K of S, $H_s(K) > 0$ if and only if K contains the support of a positive measure μ satisfying*

$$\mu(Q(\zeta,\delta)) \le C\delta^s \quad \textit{for all} \quad \zeta \in S,\ \delta > 0,$$

where C is an absolute constant.

3. Tangential boundary limits of $\mathcal{M}$-subharmonic functions

As in (1.3), for $\zeta \in S$, $\tau \ge 1$, and $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$), set

$$\mathcal{T}_{\tau,\alpha}(\zeta) = \{z \in B : |1 - \langle z, \zeta\rangle|^\tau < \alpha(1 - |z|^2)\}.$$

Also, for $0 < \rho < 1$, let

$$\mathcal{T}_{\tau,\alpha,\rho}(\zeta) = \{z \in \mathcal{T}_{\tau,\alpha}(\zeta) : \rho \le |z| < 1\}.$$

The main result of the paper is as follows:

Theorem 3.1. *Let f be a non-negative $\mathcal{M}$-subharmonic function on B satisfying*

$$\int_B (1 - |z|^2)^\gamma f^p(z)\, d\lambda(z) < \infty \tag{3.1}$$

for some $p > 0$ and $\gamma > \min\{n, pn\}$. Let $0 < s \le n$. Then for each $\tau \ge 1$, there exists a subset E_τ of S with $H_s(E_\tau) = 0$ such that for all $\zeta \in S \setminus E_\tau$ and $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$)

$$\lim_{\rho \to 1} \sup_{z \in \mathcal{T}_{\tau,\alpha,\rho}(\zeta)} (1 - |z|^2)^{\gamma - \frac{s}{\tau}} f^p(z) = 0. \tag{3.2}$$

Proof. Set $E(z) = E(z, \frac{1}{3})$. We first note that by Lemma 2.2 and inequality (2.4),

$$(1 - |z|^2)^{\gamma - \frac{s}{\tau}} f^p(z) \le C \int_{E(z)} (1 - |w|^2)^{\gamma - \frac{s}{\tau}} f^p(w)\, d\lambda(w),$$

where C is a constant depending only on n, γ, s, τ and p. Let $\tau \ge 1$, and fix $\alpha > 0$ ($\alpha > \frac{1}{2}$ if $\tau = 1$). Suppose $z \in \mathcal{T}_{\tau,\alpha}(\zeta)$. Then by Lemma 2.1, $E(z) \subset \mathcal{T}_{\tau,c}(\zeta)$ for any $c \ge \alpha 2^{\tau+1}$. Also if $|z| \ge \rho$, then by (2.4), $|w|^2 \ge 1 - 2(1 - \rho^2)$ for all $w \in E(z)$. Thus if we set $R^2 = 1 - 2(1 - \rho^2)$, $\rho \ge \sqrt{2}/2$,

$$E(z) \subset A_R = \{z \in B : R \le |z| < 1\}.$$

Therefore if $z \in \mathcal{T}_{\tau,\alpha,\rho}(\zeta)$, $E(z) \subset \mathcal{T}_{\tau,c,R}(\zeta)$, where c and R are determined as above. Thus

$$(3.3) \qquad (1-|z|^2)^{\gamma-\frac{s}{\tau}} f^p(z) \le C \int\limits_{\mathcal{T}_{\tau,c,R}(\zeta)} (1-|w|^2)^{\gamma-\frac{s}{\tau}} f^p(w)\, d\lambda(w)$$

for all $z \in \mathcal{T}_{\tau,\alpha,\rho}(\zeta)$. For $\zeta \in S$ set

$$M_{\tau,\rho}(\zeta) = \sup\{(1-|z|^2)^{\gamma-\frac{s}{\tau}} f^p(z) : z \in \mathcal{T}_{\tau,\alpha,\rho}(\zeta)\}.$$

Then by (3.3)

$$(3.4) \qquad M_{\tau,\rho}(\zeta) \le C \int\limits_{\mathcal{T}_{\tau,c,R}(\zeta)} (1-|w|^2)^{\gamma-\frac{s}{\tau}} f^p(w)\, d\lambda(w).$$

Let μ be any positive measure on S satisfying $\mu(Q(\zeta,\delta)) \le C\,\delta^s$ for all $\zeta \in S$ and $\delta > 0$. Integrating inequality (3.4) over S with respect to the measure μ gives

$$\int_S M_{\tau,\rho}(\zeta)\, d\mu(\zeta) \le C \int_S \int_{A_R} \chi_{\mathcal{T}_{\tau,c}(\zeta)}(w)(1-|w|^2)^{\gamma-\frac{s}{\tau}} f^p(w)\, d\lambda(w)\, d\mu(\zeta),$$

which by Fubini's theorem

$$\le C \int_{A_R} \left(\int_S \chi_{\widetilde{\mathcal{T}}_{\tau,c}(w)}(\zeta)\, d\mu(\zeta) \right) (1-|w|^2)^{\gamma-\frac{s}{\tau}} f^p(w)\, d\lambda(w),$$

where $\widetilde{\mathcal{T}}_{\tau,c}(w) = \{\zeta \in S : w \in \mathcal{T}_{\tau,c}(\zeta)\}$, and χ_E denotes the characteristic function of the set E. Since $|1 - \langle \frac{w}{|w|}, \zeta\rangle| \le 2|1-\langle w,\zeta\rangle|$ for all $w \ne 0$,

$$\widetilde{\mathcal{T}}_{\tau,c}(w) \subset Q(\tfrac{w}{|w|}, c'(1-|w|^2)^{1/\tau}),$$

where c' is a constant depending only on c and τ. Therefore

$$\int_S \chi_{\widetilde{\mathcal{T}}_{\tau,c}(w)}(\zeta)\, d\mu(\zeta) = \mu(\widetilde{\mathcal{T}}_{\tau,c}(w)) \le C\,(1-|w|^2)^{s/\tau}.$$

Combining the above gives

$$\int_S M_{\tau,\rho}(\zeta)\, d\mu(\zeta) \le C \int_{A_R} (1-|w|^2)^{\gamma} f^p(w)\, d\lambda(w),$$

where for $\rho \ge \sqrt{2}/2$, $R^2 = 1 - 2(1-\rho^2)$. Since f satisfies (3.1),

$$\lim_{R\to 1} \int_{A_R} (1-|w|^2)^{\gamma} f^p(w)\, d\lambda(w) = 0.$$

Thus if we let $M_\tau(\zeta) = \lim\limits_{\rho\to 1} M_{\tau,\rho}(\zeta)$, by Fatou's lemma and the above,

$$\begin{aligned}\int_S M_\tau(\zeta)\, d\mu(\zeta) &\le \lim_{\rho\to 1} \int_S M_{\tau,\rho}(\zeta)\, d\sigma(\zeta)\\ &\le C \lim_{R\to 1} \int_{A_R} (1-|w|^2)^{\gamma} f^p(w)\, d\lambda(w) = 0.\end{aligned}$$

Therefore $M_\tau(\zeta) = 0$ μ-a.e. on S. If we set $E_\tau = \{\zeta \in S : M_\tau(\zeta) > 0\}$, then $\mu(E_\tau) = 0$. Since this holds for every measure μ satisfying $\mu(Q(\zeta,\delta)) \le C\,\delta^s$, it follows that $H_s(E_\tau) = 0$, and (3.2) holds for every $\zeta \in S \setminus E_\tau$. □

Since H_n is equivalent to Lebesgue measure on S, the special case $s = n$ gives Theorem B of the Introduction as a corollary.

Corollary 3.2. *Let f be a non-negative $\mathcal{M}$-subharmonic function on B satisfying (3.1) for some $p > 0$ and $\gamma > \min\{n, pn\}$. Then for each $\tau \geq 1$ and $\alpha > 0$ ($\alpha > \frac{1}{2}$ when $\tau = 1$),*

$$\lim_{\rho \to 1} \sup_{z \in \mathcal{T}_{\tau,\alpha,\rho}(\zeta)} (1 - |z|^2)^{\gamma - \frac{n}{\tau}} f^p(z) = 0 \quad \text{for a.e.} \quad \zeta \in S.$$

From the previous corollary we also obtain the following:

Corollary 3.3. *Suppose $0 < p < 1$ and f is a non-negative $\mathcal{M}$-subharmonic function on B satisfying (3.1) for some γ, $pn < \gamma \leq n$. Then for all τ, $1 \leq \tau \leq n/\gamma$,*

$$\lim_{\substack{z \to \zeta \\ z \in \mathcal{T}_{\tau,\alpha}(\zeta)}} f(z) = 0 \quad \text{for a.e. } \zeta \in S.$$

Proof. With $\tau = n/\gamma$, by the previous corollary $f(z) \to 0$ as $z \to \zeta$, $z \in \mathcal{T}_{\tau,\alpha}(\zeta)$, at almost every $\zeta \in S$. If $1 \leq \tau' \leq \tau$, then $\mathcal{T}_{\tau',c} \subset \mathcal{T}_{\tau,c'}$, where $c' = c^{\tau'/\tau}$. Hence the result. □

Theorem 3.4. *Let f be a non-negative $\mathcal{M}$-subharmonic function on B satisfying (3.1) for some $p > 0$ and $\gamma > \min\{n, pn\}$. Then*

$$\lim_{r \to 1} (1 - r^2)^{\gamma - n} \int_S f^p(rt)\, d\sigma(t) = 0.$$

Proof. Let $\tau = 1$ and $\alpha > \frac{1}{2}$. By (3.3)

$$(1 - \rho^2)^{\gamma - n} f^p(\rho\zeta) \leq C \int_{\mathcal{T}_{1,c',R}(\zeta)} (1 - |w|^2)^{\gamma - n} f^p(w)\, d\lambda(w)$$

for all $\zeta \in S$ and ρ sufficiently close to 1. As in the previous theorem, by integrating over S,

$$(1 - \rho^2)^{\gamma - n} \int_S f^p(\rho\zeta)\, d\sigma(\zeta) \leq C \int_{A_R} (1 - |w|^2)^{\gamma} f^p(w)\, d\lambda(w),$$

where $R^2 = 1 - 2(1 - \rho^2)$, from which the result follows. □

Remark. If $p \geq 1$, then the conclusion of Theorem 3.4 follows immediately from the fact that f^p is $\mathcal{M}$-subharmonic on B, and thus $\int_S f^p(rt) d\sigma(t)$ is a nondecreasing function of r, $0 < r < 1$. As a consequence, if $0 < R < 1$, a straightforward argument gives

$$\int_{A_R} (1 - |z|^2)^{\gamma} f^p(z)\, d\lambda(z) \geq C(1 - R^2)^{\gamma - n} \int_S f^p(Rt)\, d\sigma(t),$$

from which the conclusion follows.

4. Non-integrability of $\mathcal{M}$-subharmonic functions

In this section we consider integrability criteria for non-negative $\mathcal{M}$-subharmonic functions on B. The results of this section are motivated by the following question: Given p, $0 < p < \infty$, for what values of γ does there exist a non-negative $\mathcal{M}$-subharmonic function on B such that the integral in (1.4) is finite? For non-negative subharmonic functions on domains in $\mathbb{R}^n$ this problem was considered by N. Suzuki in [SZ].

For convenience, if $\gamma \in \mathbb{R}$ and $0 < p < \infty$, let L^p_γ denote the set of measurable functions f on B for which

$$\int_B (1-|z|^2)^\gamma |f(z)|^p \, d\lambda(z) < \infty. \tag{4.1}$$

If $\gamma > n$, then the measure $(1-|z|^2)^\gamma d\lambda(z)$ is a finite measure on B. Thus every bounded $\mathcal{M}$-subharmonic or $\mathcal{M}$-harmonic function on B is in L^p_γ for all p, $0 < p < \infty$. In particular, if f is a bounded holomorphic function on B, then $|f|$ is a non-negative plurisubharmonic, and thus $\mathcal{M}$-subharmonic, function on B satisfying (4.1) for all $\gamma > n$ and $0 < p < \infty$. Conversely, if $p \geq 1$, then, as we will prove in Theorem 4.1, the only non-negative $\mathcal{M}$-subharmonic function $f \in L^p_\gamma$ for some $\gamma \leq n$ is the zero function.

If $0 < p < 1$, the results are somewhat different. If f is holomorphic on B and $f \in L^p_\gamma$ for some $\gamma \leq n$ and $0 < p < 1$, then $f(z) = 0$ for all $z \in B$. This is due to the fact that $|f(z)|^p$ is $\mathcal{M}$-subharmonic for all $p > 0$. For $\mathcal{M}$-subharmonic functions however, when $0 < p < 1$, there exist values of $\gamma \leq n$ and $\mathcal{M}$-subharmonic functions f, $f \not\equiv 0$, with $f \in L^p_\gamma$. Examples of such functions will be given in Examples 4.3 – 4.5.

Our first result justifies the hypothesis $\gamma > \min\{n, pn\}$ of Theorem 3.1.

Theorem 4.1. *(a) Let $0 < p < \infty$. If f is a non-negative $\mathcal{M}$-subharmonic function on B with $f \in L^p_\gamma$ for some $\gamma \leq \min\{n, pn\}$, then $f \equiv 0$.*

(b) If $0 < p < 1$ and f is an $\mathcal{M}$-subharmonic function on B with $f \in L^p_\gamma$ for some $\gamma \leq \min\{pn, (1-p)n\}$, then $f \equiv 0$.

Proof. (a) Suppose first that $p \geq 1$. Then f^p is also $\mathcal{M}$-subharmonic on B. By the $\mathcal{M}$-invariance of λ it is clear that $f \in L^p_\gamma$ if and only if $f \circ \varphi_a \in L^p_\gamma$ for all $a \in B$. If $0 < R < 1$, then since f^p is $\mathcal{M}$-subharmonic,

$$\begin{aligned}
&\int_{A_R} (1-|w|^2)^\gamma f^p(\varphi_a(w)) \, d\lambda(w) \\
&\qquad \geq 2n \int_R^{(1+R)/2} r^{2n-1}(1-r^2)^{\gamma-n-1} \int_S f^p(\varphi_a(rt)) \, d\sigma(t) \, dr \\
&\qquad \geq C\,(1-R^2)^{\gamma-n} \, f^p(a).
\end{aligned}$$

Thus

$$0 \leq f^p(a) \leq C\,(1-R^2)^{n-\gamma} \int_{A_R} (1-|w|^2)^\gamma f^p(w) \, d\lambda(w).$$

If $f \in L^p_\gamma$ for $\gamma \leq n$, then the term on the right converges to 0 as $R \to 1$. Hence $f^p(a) = 0$ for all $a \in B$.

Suppose $0 < p < 1$ and $f \in L^p_\gamma$. By inequality (2.4) and Lemma 2.2, for $0 < r < 1$ and $t \in S$,

$$\begin{aligned}
f^p(rt) &\leq C \int_{E(rt)} f^p(w) \, d\lambda(w) \\
&\leq C\,(1-r^2)^{-\gamma} \int_{E(rt)} (1-|w|^2)^\gamma f^p(w) \, d\lambda(w) \leq C'\,(1-r^2)^{-\gamma}
\end{aligned}$$

for some finite constant C'. Hence

$$\int_S f^p(rt)\,d\sigma(t) = \int_S f(rt)(f(rt))^{p-1} d\sigma(t) \geq C\,(1-r^2)^{-\frac{\gamma}{p}(p-1)} \int_S f(rt)\,d\sigma(t) \geq C\,(1-r^2)^{-\gamma+\frac{\gamma}{p}} f(0).$$

Therefore by integration in polar coordinates,

$$\int_{A_R} (1-|z|^2)^\gamma f^p(z)\,d\lambda(z) = 2n \int_R^1 r^{2n-1}(1-r^2)^{\gamma-n-1} \int_S f^p(rt)\,d\sigma(t)\,dr \geq C\,f(0) \int_R^1 r^{2n-1}(1-r^2)^{\frac{\gamma}{p}-n-1}\,dr = +\infty$$

for all $\gamma \leq pn$. Thus the only non-negative $\mathcal{M}$-subharmonic function satisfying (4.1) for $\gamma \leq pn$ is the zero function.

(b) Suppose $0 < p < 1$ and f is $\mathcal{M}$-subharmonic on B with $f \in L^p_\gamma$ for some $\gamma \leq \min\{pn, (1-p)n\}$. Let $f^+(z) = \max\{f(z), 0\}$. Then f^+ is a non-negative $\mathcal{M}$-subharmonic function on B with $f^+ \in L^p_\gamma$ for some $\gamma \leq pn$. Thus by the first part of the theorem $f^+ \equiv 0$. Thus $|f| = -f$, which is a non-negative $\mathcal{M}$-superharmonic function on B. By the Riesz decomposition theorem [ST2, Corollary 6.11]; [UL, Theorem 2.16],

$$|f(z)| = \int_B G(z,w)\,du(w) + \int_S P(z,t)\,d\nu(t),$$

where ν is a finite measure on S and μ is a regular Borel measure on B satisfying

$$\int_B (1-|w|^2)^n\,d\mu(w) < \infty.$$

Since $P(z,t) \geq c_1(1-|z|^2)^n$ and $G(z,w) \geq c_2(1-|z|^2)^n(1-|w|^2)^n$ for positive constants c_1 and c_2,

$$|f(z)| \geq c_1(1-|z|^2)^n \int_B (1-|w|^2)^n\,d\mu(w) + c_2(1-|z|^2)^n \nu(S) \geq C\,(1-|z|^2)^n,$$

where C is positive unless both μ and ν are the zero measures; i.e., $f \equiv 0$. Hence if f is not identically zero,

$$\int_B (1-|z|^2)^\gamma |f(z)|^p d\lambda(z) \geq C \int_B (1-|z|^2)^{\gamma+pn} d\lambda(z) = +\infty$$

for any γ satisfying $\gamma + pn \leq n$; i.e., $\gamma \leq n(1-p)$. □

Remarks. (a) If f is holomorphic on B, then $|f|^p$ is $\mathcal{M}$-subharmonic for all $p > 0$. Thus the same argument as used in (a) (for $p \geq 1$) proves that if $f \in L^p_\gamma$ for some $p > 0$ and $\gamma \leq n$, then $f(z) = 0$ for all $z \in B$.

(b) The proof of (b) also shows that if f is a non-negative $\mathcal{M}$-superharmonic function on B with $f \in L^p_\gamma$ for some p, $0 < p < 1$, and $\gamma \leq n(1-p)$, then $f(z) = 0$ for all $z \in B$.

For a non-negative $\mathcal{M}$-harmonic function h on B, if $h \in L^p_\gamma$ for some $\gamma \leq \min\{pn, n\}$, then by Theorem 4.1 we must have $h(z) = 0$ for all $z \in B$. However, when $0 < p < 1$, we have the following stronger result.

Theorem 4.2. *Let $0 < p < 1$. If h is a non-negative $\mathcal{M}$-harmonic function on B with $h \in L^p_\gamma$ for some $\gamma \le \max\{pn, (1-p)n\}$, then $h \equiv 0$.*

Proof. If $\frac{1}{2} \le p < 1$, then $\max\{pn, (1-p)n\} = pn$, and thus the conclusion follows by Theorem 4.1. Suppose now that $0 < p < \frac{1}{2}$. Since h is a non-negative $\mathcal{M}$-harmonic function on B,

$$h(z) = \int_S P(z,t)\, d\nu(t),$$

where ν is a finite measure on S. But then

$$\int_B (1-|z|^2)^\gamma h^p(z)\, d\lambda(z) \ge C\,\nu(S)^p \int_B (1-|z|^2)^{\gamma+pn} d\lambda(z) = +\infty$$

for any $\gamma \le (1-p)n$ unless $\nu(S) = 0$; i.e., $h \equiv 0$. □

Example 4.3. In this example we show that the conclusion of Theorem 4.2 is best possible. As in (2.3) let P be the invariant Poisson kernel on B. Set

$$h(z) = P(z, e_1) = \frac{(1-|z|^2)^n}{|1-z_1|^{2n}},$$

where $e_1 = (1, 0, ..., 0)$. Then h is a non-negative $\mathcal{M}$-harmonic function on B, and

$$\int_B (1-|z|^2)^\gamma h^p(z)\, d\lambda(z) = 2n \int_0^1 r^{2n-1}(1-r^2)^{\gamma+pn-n-1} \int_S \frac{d\sigma(t)}{|1-rt_1|^{2pn}}\, dr.$$

By [RU, Proposition 1.4.10]

$$\int_S \frac{d\sigma(t)}{|1-rt_1|^{2pn}} \le C \begin{cases} (1-r^2)^{n-2pn}, & \frac{1}{2} < p < 1, \\ -\log(1-r^2), & p = \frac{1}{2}, \\ \quad 1, & 0 < p < \frac{1}{2}. \end{cases} \tag{4.2}$$

From this it now follows that for $0 < p < 1$, $h \in L^p_\gamma$ for all γ satisfying $\gamma > \max\{pn, (1-p)n\}$. This example also shows that the conclusion of Theorem 4.1(b) is best possible.

Example 4.4. In this example we show that when $n = 1$ and $0 < p < 1$, then for each $\gamma > p$ there exists a non-negative subharmonic function f on $D = \{z \in \mathbb{C} : |z| < 1\}$ with $f \not\equiv 0$, such that $f \in L^p_\gamma$. For $0 < \beta < \frac{\pi}{2}$, consider the angular region S_β with vertex at 1 defined by

$$S_\beta = \{z \in D : |\arg(1-z)| < \beta,\ |1-z| < \cos\beta\}.$$

The set S_β is simply a truncated Stolz's domain with $S_\beta \subset \Gamma_\alpha(1)$ where $\alpha = 1\big/\cos\beta$. Let φ_β be a conformal mapping of S_β onto D, mapping the boundary of S_β onto the boundary of D with $\varphi_\beta(1) = 1$ Consider the function f_β defined on D by

$$f_\beta(z) = \begin{cases} P(\varphi_\beta(z), 1), & z \in S_\beta, \\ \quad 0, & z \in D \setminus S_\beta, \end{cases}$$

where $P(w,1) = \dfrac{1-|w|^2}{|1-w|^2}$ is the Poisson kernel on D. Thus $P(\varphi_\beta(z), 1)$ is harmonic on S_β and 0 on $\partial S_\beta \setminus \{1\}$. Hence the function f_β is subharmonic on D.

As in [MA, Lemma 2.2], there exists a non-zero holomorphic function h defined on a neighborhood N of 1 such that

$$1-\varphi_\beta(z)=(1-z)^b h(z)$$

for all $z\in N\cap S_\beta$, where $b=\frac{\pi}{2\beta}$. Write $b=1+\epsilon(\beta)$, where $\epsilon(\beta)\to 0$ as $\beta\to\frac{1}{2}\pi$. Thus for all $z\in S_\beta$,

$$f_\beta(z)\le P(\varphi_\beta(z),1)\le C\,\frac{1}{|1-z|^{1+\epsilon(\beta)}}.$$

Hence,

$$\int_D (1-|z|^2)^\gamma f_\beta^p(z)\,d\lambda(z)\le C\int_0^1(1-r^2)^{\gamma-2}\int_0^{2\pi}\frac{\chi_{S_\beta}(re^{i\theta})}{|1-re^{i\theta}|^{p+p\epsilon(\beta)}}\,d\theta\,rdr.$$

But

$$\int_0^{2\pi}\frac{\chi_{S_\beta}(re^{i\theta})}{|1-re^{i\theta}|^{p+p\epsilon(\beta)}}\,d\theta\le C\,(1-r^2)^{-p-p\epsilon(\beta)}\sigma(\widetilde{S}_\beta(r))\le C_\beta(1-r^2)^{1-p-p\epsilon(\beta)}.$$

In the above, $\widetilde{S}_\beta(r)=\{\zeta\in S: r\zeta\in S_\beta\}$. Therefore,

$$\int_D (1-|z|^2)^\gamma f_\beta^p(z)\,d\lambda(z)\le C_\beta\int_0^1(1-r^2)^{\gamma-p-p\epsilon(\beta)-1}r\,dr.$$

If $\gamma>p$, then we can choose β sufficiently close to $\frac{1}{2}\pi$ such that $\gamma-p-p\epsilon(\beta)>0$, in which case the above integral is finite.

For the case $n=1$ another example can also be found in [SZ]. Even though it is conjectured that for $n\ge 2$ the conclusion of Theorem 4.1 is also sharp for all p, $0<p<1$, we have not been able to construct an appropriate example at this time. The following example does however show that for $n\ge 2$ and $\frac{1}{2}\le p<1$, the conclusion of Theorem 4.1 is best possible.

Example 4.5. In this example we show that when $n>1$ and $0<p<1$, then for each $\gamma>\max\{pn,\frac{n}{2}\}$ there exists a positive $\mathcal{M}$-subharmonic function $f\in L^p_\gamma$.

If $\frac{1}{2}\le p<1$ and $\gamma>pn$, choose $\beta>1$ such that $\gamma>\beta pn$. If $0<p<\frac{1}{2}$ and $\gamma>\frac{n}{2}>pn$, then choose $\beta>1$ such that

$$\frac{\gamma}{pn}>\beta>\frac{1}{2p}.$$

Thus in both cases $\gamma>\beta pn$ and $\beta p>\frac{1}{2}$. Let $f_\beta(z)=P^\beta(z,e_1)$. Since $\beta>1$, f_β is $\mathcal{M}$-subharmonic on B. For this function

$$\int_B(1-|z|^2)^\gamma f_\beta^p(z)\,d\lambda(z)\le 2n\int_0^1 r^{2n-1}(1-r^2)^{\gamma+\beta pn-n-1}\int_S\frac{d\sigma(t)}{|1-rt_1|^{2n\beta p}}\,dr,$$

which since $\beta p>\frac{1}{2}$, by (4.2)

$$\le C\int_0^1(1-r^2)^{\gamma-\beta pn-1}r^{2n-1}dr.$$

Since $\gamma>\beta pn$, this last integral is finite. Thus $f_\beta\in L^p_\gamma$.

References

[BC] R. D. Berman and W. S. Cohn, *Littlewood theorems for limits and growth of potentials along level sets of Hölder continuous functions*, Amer. J. of Math. **114** (1991), 185–227. MR **93c:**31006

[CA] G. T. Cargo, *Angular and tangential limits of Blaschke products and their successive derivatives*, Canad. J. Math. **14** (1962), 334–348. MR **25:**204

[CO] W. S. Cohn, *Non-isotropic Hausdorff measure and exceptional sets for holomorphic Sobolev functions*, Illinois J. Math. **33** (1989), 673–690. MR **90j:**32008

[FS] C. Fefferman and S. Stein, *H^p spaces of several variables*, Acta Math **129** (1972), 137–193. MR **56:**6263

[GE] F. W. Gehring, *On the radial order of subharmonic functions*, Jour. Math. Soc. Japan **9** (1957), 77–79. MR **19:**131e

[HSY] K. T. Hahn, M. Stoll, and H. Youssfi, *Invariant potentials and tangential boundary behavior of $\mathcal{M}$-subharmonic functions in the unit ball*, Complex Variables **28** (1995), 67–96.

[HY] K. T. Hahn and H. Youssfi, *Tangential boundary behavior of $\mathcal{M}-$harmonic Besov functions in the unit ball*, J. Math. Analysis and Appl. **175** (1993), 206–221. MR **94c:**32006

[HL] G. H. Hardy and J. E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math. **167** (1932), 403–423.

[KI1] J. R. Kinney, *Boundary behavior of Blaschke products in the unit circle*, Proc. Amer. Math. Soc. **12** (1961), 484–488. MR **23:**A2533

[KI2] ———, *Tangential limits of functions of the class S_α*, Proc. Amer. Math. Soc. **14** (1963), 68–70. MR **26:**1466

[KO] P. Koosis, *Introduction to H^p Spaces*, London Math. Soc. Lecture Notes **40** (1980). MR **81c:**30062

[MA] B. D. MacCluer, *Compact composition operators on $H^p(B_n)$*, Michigan Math. J. **32** (1985), 237–248. MR **86g:**47037

[NRS] A. Nagel, W. Rudin, and J. H. Shapiro, *Tangential boundary behavior of functions in Dirichlet-type spaces*, Annals of Math. **116** (1982), 331–360. MR **84a:**31002

[PA] M. Pavlovic, *Inequalities for the gradient of eigenfunctions of the invariant laplacian in the unit ball*, Indag. Mathem., N.S. **2** (1991), 89–98. MR **92d:**32008

[RU] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, New York, 1980. MR **82i:**32002

[RZ] L. Rzepecki, *Boundary behavior of non-isotropic potentials in the unit ball of $\mathbb{C}^n$*, Ph. D. Dissertation, University of South Carolina (1995).

[ST1] M. Stoll, *Tangential boundary limits of invariant potentials in the unit ball of $\mathbb{C}^n$*, J. Math. Analysis & Appl. **177** (1993), 553–571. MR **94h:**32020

[ST2] ———, *Invariant Potential Theory in the Unit Ball of $\mathbb{C}^n$*, London Math. Soc. Lect. Note Series **199** (1994). MR **96f:**31011

[ST3] ———, *Non-isotropic Hausdorff capacity of exceptional sets of invariant potentials*, Potential Analysis **4** (1995), 141–155. MR **96b:**31011

[SU] J. Sueiro, *Tangential boundary limits and exceptional sets for holomorphic functions in Dirichlet-type spaces*, Math. Ann. **286** (1990), 661–678. MR **91b:**32008

[SZ] N. Suzuki, *Nonintegrability of harmonic functions in a domain*, Japan J. Math. **16** (1990), 269–278. MR **91m:**31003

[TS] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea Publ. Co., New York, N.Y., 1975. MR **54:**2990

[TW] J. B. Twomey, *Tangential limits for certain classes of analytic functions*, Mathematika **36** (1989), 39–49. MR **91b:**30100

[UL] D. Ullrich, *Radial limits of $\mathcal{M}$-subharmonic functions*, Trans. Amer. Math. Soc. **292** (1985), 501–518. MR **87a:**31007

[ZH] S. Zhao, *On the weighted L^p-integrability of nonnegative $\mathcal{M}$-superharmonic functions*, Proc. Amer. Math. Soc. **115** (3) (1992), 677–685. MR **92i:**31005

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208

E-mail address: `stoll@math.sc.edu`